

SUPERCONGRUENCES RELATED TO ${}_3F_2(1)$ INVOLVING HARMONIC NUMBERS

ROBERTO TAURASO

ABSTRACT. We show various supercongruences for truncated series which involve central binomial coefficients and harmonic numbers. The corresponding infinite series are also evaluated.

1. INTRODUCTION

In 1997, Van Hamme [21] established the p -adic analogs of several Ramanujan type series. For one of them, the series labeled (H.1),

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{64^k} = {}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] = \frac{\pi}{\Gamma^4(\frac{3}{4})}, \quad (1)$$

the modulo p^2 congruence (H.2) for the truncated version has been recently improved by Long and Ramakrishna in [9, Theorem 3],

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv_{p^3} \begin{cases} -\Gamma_p^4(\frac{1}{4}) & \text{if } p \equiv_4 1, \\ -\frac{p^2}{16} \Gamma_p^4(\frac{1}{4}) & \text{if } p \equiv_4 3. \end{cases} \quad (2)$$

where p is any prime greater than 3 (we use the notation $a \equiv_m b$ to mean $a \equiv b \pmod{m}$).

In this paper we will investigate the series and the corresponding partial sums where the terms have one of the following forms

$$\binom{2k}{k}^3 \frac{H_k}{64^k}, \quad \binom{2k}{k}^3 \frac{H_k^{(2)}}{64^k}, \quad \binom{2k}{k}^3 \frac{O_k}{64^k}, \quad \binom{2k}{k}^3 \frac{O_k^{(2)}}{64^k}.$$

Here $H_k^{(r)}$ denotes the k -th generalized harmonic number of order r and $O_k^{(r)}$ is the sum with odd denominators,

$$H_k^{(r)} = \sum_{j=1}^k \frac{1}{j^r} \quad \text{and} \quad O_k^{(r)} = \sum_{j=1}^k \frac{1}{(2j-1)^r}$$

where we adopt the convention that $H_k = H_k^{(1)}$ and $O_k = O_k^{(1)}$.

Date: January 31, 2017.

2010 Mathematics Subject Classification. 11A07, 33C20, 11S80, 33B15, 11B65.

Key words and phrases. Supercongruences, hypergeometric series, harmonic numbers, p -adic Gamma function.

The main results are presented in Section 3 (evaluations of the infinite series) and Section 5 (congruences for the truncated series). For example we show that

$$\sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{O_k}{64^k} = \frac{\pi^2}{6\Gamma^4(\frac{3}{4})}, \quad \text{and} \quad \sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{O_k}{64^k} \equiv_{p^2} \begin{cases} 0 & \text{if } p \equiv_4 1, \\ -\frac{p}{12}\Gamma_p^4\left(\frac{1}{4}\right) & \text{if } p \equiv_4 3. \end{cases}$$

The correspondence between the right-hand sides of the infinite series and the finite sum is particularly striking for the appearance of the classic Gamma function and the p -adic analog.

2. SIMILAR RESULTS OF LOWER DEGREE

Before dealing with the main issue, we are going to take a look to similar sums already in the literature, where the central binomial coefficient is raised to a power less than 3. Assume that p is a prime greater than 3. For $n \geq 1$, we have

$$\sum_{k=1}^{n-1} \binom{2k}{k} \frac{H_k}{4^k} = \binom{2n}{n} \frac{2n(H_{n-1} - 2)}{4^n} + 2.$$

Thus, by $n = p$, we obtain (see [19, (1.10)] for the modulo p^3 version)

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k}{4^k} \equiv_{p^4} 2 - 2p + 4p^2 q_p(2) - 6p^3 q_p^2(2) - \frac{1}{3}p^3 B_{p-3},$$

where $q_p(a) = \frac{a^{p-1}-1}{p}$ is the Fermat quotient and we used the Wolstenholme's theorem $\binom{2p}{p} \equiv_{p^3} 2$, and the congruences

$$H_{p-1} \equiv_{p^3} -\frac{1}{3}p^2 B_{p-3}, \quad 4^{p-1} \equiv_{p^3} 1 + 2p q_p(2) + p^2 q_p^2(2) \quad (3)$$

(for the first one we can refer to [14, Theorem 5.1 (a)]). Moreover, the identity

$$\sum_{k=1}^{n-1} \binom{2k}{k} \frac{H_k^{(2)}}{4^k} = \binom{2n}{n} \frac{2n H_{n-1}^{(2)}}{4^n} - 2 \sum_{k=1}^{n-1} \frac{\binom{2k}{k}}{k 4^k}$$

implies (see [19, (1.11)] for the modulo p version)

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k^{(2)}}{4^k} \equiv_{p^3} -4q_p(2) + 2p q_p^2(2) - \frac{4}{3}p^2 q_p^3(2) - \frac{1}{2}p^2 B_{p-3},$$

where we employed the congruence established in [20, Theorem 1.1],

$$\sum_{k=1}^{n-1} \frac{\binom{2k}{k}}{k 4^k} \equiv_{p^3} -H_{\frac{p-1}{2}}$$

and

$$H_{p-1}^{(2)} \equiv_{p^2} \frac{2}{3}p B_{p-3}, \quad H_{\frac{p-1}{2}} \equiv_{p^3} -2q_p(2) + p q_p^2(2) - \frac{2}{3}p^2 q_p^3(2) - \frac{7}{12}p^2 B_{p-3} \quad (4)$$

given in [14, Theorem 5.1 (a)] and [14, Theorem 5.2 (c)] respectively.

As regards the squared case, the identities [12, (2.4) and (2.8)]

$$\begin{aligned} \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k &= 2(-1)^n H_n, \\ \sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} (-1)^k H_k^{(2)} &= 2(-1)^{n+1} \sum_{k=1}^n \frac{(-1)^k}{k^2}, \end{aligned}$$

and the congruence for $0 \leq k \leq n = (p-1)/2$ (note that p divides $\binom{2k}{k}$ for $n < k < p$)

$$\binom{n}{k} \binom{n+k}{k} (-1)^k = \binom{2k}{k} \frac{\prod_{j=1}^k ((2j-1)^2 - p^2)}{4^k (2k)!} \equiv_{p^2} \frac{\binom{2k}{k}^2}{16^k} \quad (5)$$

imply [19, Theorem 4.1] (see also [17, Theorems 1.1 and 1.2] for a more general p^2 -congruence)

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{2k}{k}^2 \frac{H_k}{16^k} &\equiv_{p^2} (-1)^{\frac{p+1}{2}} (4q_p(2) - 2pq_p^2(2)), \\ \sum_{k=1}^{p-1} \binom{2k}{k}^2 \frac{H_k^{(2)}}{16^k} &\equiv_{p^2} -8E_{p-3} + 4E_{2p-4}, \end{aligned}$$

where we also used

$$H_{\frac{p-1}{2}}^{(2)} \equiv_{p^2} \frac{7}{3} p B_{p-3}, \quad H_{[\frac{p}{4}]}^{(2)} \equiv_{p^2} (-1)^{\frac{p-1}{2}} (8E_{p-3} - 4E_{2p-4}) + \frac{14}{3} p B_{p-3} \quad (6)$$

given in [14, Corollary 5.2], [15, Corollary 3.8] and

$$\sum_{k=1}^n \frac{(-1)^k}{k^2} = \frac{1}{2} H_{[\frac{p}{4}]}^{(2)} - H_{\frac{p-1}{2}}^{(2)} \equiv_{p^2} (-1)^{\frac{p-1}{2}} (8E_{p-3} - 4E_{2p-4}). \quad (7)$$

3. EVALUATIONS OF THE INFINITE SERIES

The generalized hypergeometric function is defined as

$${}_rF_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_r)_k}{(b_1)_k (b_2)_k \dots (b_s)_k} \cdot \frac{z^k}{k!}$$

where $(x)_k = x(x+1) \dots (x+k-1)$ for $k \geq 1$ and $(x)_0 = 1$ is the Pochhammer symbol and a_i, b_j and z are complex numbers with none of the b_j being negative integers or zero. We recall some well-known hypergeometric identities:

i) Dixon's theorem [1, p.13]

$${}_3F_2 \left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}; 1 \right] = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}, \quad (8)$$

ii) Whipple's theorem [1, p.16]

$${}_3F_2 \left[\begin{matrix} a, 1-a, c \\ e, 1+2c-e \end{matrix}; 1 \right] = \frac{\pi 2^{1-2c} \Gamma(e) \Gamma(1+2c-e)}{\Gamma(\frac{a+c}{2}) \Gamma(\frac{1-a+c}{2}) \Gamma(1+c-\frac{a+c}{2}) \Gamma(1+c-\frac{1-a+c}{2})}. \quad (9)$$

In the next theorem we evaluate four specific series.

Theorem 1. *We have that*

$$\sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{H_k}{64^k} = \frac{2\pi(\pi - 3\ln 2)}{3\Gamma^4(\frac{3}{4})}, \quad \sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{O_k}{64^k} = \frac{\pi^2}{6\Gamma^4(\frac{3}{4})}, \quad (10)$$

$$\sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{H_k^{(2)}}{64^k} = \frac{\pi(12G - \pi^2)}{3\Gamma^4(\frac{3}{4})}, \quad \sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{O_k^{(2)}}{64^k} = \frac{\pi(\pi^2 - 8G)}{8\Gamma^4(\frac{3}{4})}. \quad (11)$$

where $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ is the Catalan's constant.

Proof. Let

$$H_k^{(r)}(x) = \sum_{j=0}^{k-1} \frac{1}{(x+j)^r}.$$

Then

$$\frac{d}{dx} ((x)_k) = (x)_k \cdot H_k(x) \quad \text{and} \quad \frac{d}{dx} (H_k^{(r)}(x)) = -r H_k^{(r+1)}(x).$$

For (10), let $a = b = 1/2$ in (8), then

$$\left. \frac{\partial}{\partial c} \left({}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, c \\ 1, \frac{3}{2} - c \end{matrix}; 1 \right] \right) \right|_{c=\frac{1}{2}} = \sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{2O_k + H_k}{64^k}.$$

By setting $b = c = 1/2$ in (8), we get

$$\left. \frac{\partial}{\partial a} \left({}_3F_2 \left[\begin{matrix} a, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} + a, \frac{1}{2} + a \end{matrix}; 1 \right] \right) \right|_{a=\frac{1}{2}} = \sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{2O_k - 2H_k}{64^k}.$$

On the other hand, by differentiating the right-hand side of (8) and (9) and by using

$$\frac{d}{dx} (\Gamma(x)) = \Gamma(x) \cdot \Psi(x) \quad \text{and} \quad \frac{d}{dx} (\Psi^{(r)}(x)) = \Psi^{(r+1)}(x)$$

where $\Psi^{(r)}$ is the polygamma function of order r (with $\Psi^{(0)} = \Psi$), we obtain

$$\left. \frac{\partial}{\partial c} \left({}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, c \\ 1, \frac{3}{2} - c \end{matrix}; 1 \right] \right) \right|_{c=\frac{1}{2}} = {}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] \cdot \left(-\Psi(1) - \Psi\left(\frac{1}{4}\right) + \Psi\left(\frac{3}{4}\right) + \Psi\left(\frac{1}{2}\right) \right),$$

and

$$\left. \frac{\partial}{\partial a} \left({}_3F_2 \left[\begin{matrix} a, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} + a, \frac{1}{2} + a \end{matrix}; 1 \right] \right) \right|_{a=\frac{1}{2}} = {}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] \cdot \left(2\Psi(1) - 2\Psi\left(\frac{3}{4}\right) - 2\ln(2) \right).$$

By considering a suitable linear combination of the previous two identities, the special values

$$\Psi\left(\frac{1}{2}\right) - \Psi(1) = -\ln 4, \quad \Psi\left(\frac{1}{4}\right) - \Psi(1) = -\ln 8 - \frac{\pi}{2}, \quad \Psi\left(\frac{3}{4}\right) - \Psi(1) = -\ln 8 + \frac{\pi}{2}.$$

yield immediately (10).

Let $a = c = 1/2$ in (9), then

$$\left. \frac{\partial^2}{\partial e^2} \left({}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ e, 2-e \end{matrix}; 1 \right] \right) \right|_{e=1} = \sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{2H_k^{(2)}}{64^k}.$$

Moreover, for $c = 1/2$, $e = 1$ in (9), we find

$$\left. \frac{\partial^2}{\partial a^2} \left({}_3F_2 \left[\begin{matrix} a, 1-a, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] \right) \right|_{a=\frac{1}{2}} = \sum_{k=1}^{\infty} \binom{2k}{k}^3 \frac{-8O_k^{(2)}}{64^k}.$$

On the right-hand side, we have

$$\left. \frac{\partial^2}{\partial e^2} \left({}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ e, 2-e \end{matrix}; 1 \right] \right) \right|_{e=1} = {}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] \cdot \left(\frac{\pi^2}{3} - \Psi_1 \left(\frac{3}{4} \right) \right),$$

and

$$\left. \frac{\partial^2}{\partial a^2} \left({}_3F_2 \left[\begin{matrix} a, 1-a, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] \right) \right|_{a=\frac{1}{2}} = {}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right] \cdot \left(\frac{1}{2} \Psi_1 \left(\frac{1}{4} \right) - \frac{1}{2} \Psi_1 \left(\frac{3}{4} \right) - \pi^2 \right).$$

As before, by combining the results and by using the special values

$$\Psi_1 \left(\frac{1}{2} \pm \frac{1}{4} \right) = \pi^2 \mp 8G$$

the conclusion (11) easily follows. \square

4. CONGRUENCES FOR THE TRUNCATED SERIES - PRELIMINARY RESULTS

If n is an odd integer, by replacing k with $(n - k)$ is easy to see that

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = 0 \quad \text{and} \quad \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_k H_{n-k} = 0. \quad (12)$$

The next lemma follows from [2, Theorem 1].

Lemma 1. *For any non-negative odd integer $n = 2m + 1$, we have*

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_k = -\frac{c_m}{6}, \quad (13)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 (3H_k^2 + H_k^{(2)}) = \frac{c_m}{2} (H_m - 4H_{2m+1} - H_{3m+2} + 2H_{6m+4}). \quad (14)$$

where $c_m = \frac{(-1)^m (6m+3)! (m!)^3}{(3m+1)! ((2m+1)!)^3}$.

The next lemma establishes some identities involving the harmonic numbers that we will need later on.

Lemma 2. *For any non-negative integer n , we have*

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} \frac{H_k^{(2)}}{(-4)^k} = \begin{cases} \left(\frac{n}{2}\right)^2 \cdot \frac{\sum_{k=1}^n \frac{(-1)^k}{k^2}}{4^n} & \text{if } n \equiv_2 0, \\ \left(\frac{n-1}{2}\right)^{-2} \cdot \frac{-4^{n-1}}{n^2} & \text{if } n \equiv_2 1. \end{cases} \quad (15)$$

Moreover, for any even integer,

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} \frac{H_k}{(-4)^k} = \left(\frac{n}{2}\right)^2 \frac{H_n}{4^n}, \quad (16)$$

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} \frac{H_{2k}}{(-4)^k} = \left(\frac{n}{2}\right)^2 \frac{H_n}{2 \cdot 4^n}. \quad (17)$$

Proof. For $n = 2m$, let

$$F(m, k) = \binom{2m}{k} \binom{2m+k}{k} \binom{2k}{k} \binom{2m}{m}^{-2} (-4)^{2m-k}$$

then by Wilf-Zeilberger method we find

$$G(m, k) = -\frac{2(4m+3)k^2}{(2m+1)^3} \binom{2m+1}{k-1} \binom{2m+k}{k} \binom{2k}{k} \binom{2m}{m}^{-2} (-4)^{2m-k}$$

such that

$$F(m+1, k) - F(m, k) = G(m, k+1) - G(m, k).$$

Let $S(m) = \sum_{k \geq 1} F(m, k) H_k^{(2)}$ then, by summation by parts (see [5] for a similar approach), we have

$$\begin{aligned} S(m+1) - S(m) &= \sum_{k \geq 0} (G(m, k+1) - G(m, k)) H_k^{(2)} \\ &= -\sum_{k \geq 0} \frac{G(m, k+1)}{(k+1)^2} = -\sum_{k \geq 1} \frac{G(m, k)}{k^2} \\ &= -\frac{1}{(2m+1)^2} + \frac{1}{(2m+2)^2}. \end{aligned}$$

The other identities can be obtained in a similar way. □

The Morita's p -adic Gamma function Γ_p is defined as the continuous extension to the set of all p -adic integers \mathbb{Z}_p of the sequence

$$n \rightarrow (-1)^n \prod_{\substack{0 \leq k < n \\ (k,p)=1}} k$$

where p is an odd prime and $n > 1$ is an integer (see [13, Chapter 7] for a detailed introduction to Γ_p). If $x \in \mathbb{Z}_p$ then $\Gamma_p(0) = 1$ and

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } |x|_p = 1, \\ -\Gamma_p(x) & \text{if } |x|_p < 1, \end{cases}$$

where $|\cdot|_p$ denotes the p -adic norm. By [9, Theorem 14], for all $a, b \in \mathbb{Z}_p$,

$$\Gamma_p(a+bp) \equiv_{p^2} \Gamma_p(a)(1+G_1(a)bp) \quad (18)$$

where $G_1(a) = \Gamma'_p(a)/\Gamma_p(a) \in \mathbb{Z}_p$. Moreover

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{s_p(x)} \quad (19)$$

where $s_p(x)$ is the integer in $\{1, 2, \dots, p\}$ such that $s_p(x) \equiv_p x$. The above formula is the p -adic analog of the classic reflection formula for the classic Gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Lemma 3. *For any prime $p > 3$,*

$$\frac{\binom{2m}{m}^2}{16^m} \equiv_{p^2} \begin{cases} -\Gamma_p^4\left(\frac{1}{4}\right) & \text{if } p \equiv_4 1, \\ 16\Gamma_p^{-4}\left(\frac{1}{4}\right)(1+2p) & \text{if } p \equiv_4 3, \end{cases} \quad (20)$$

where $m = \lfloor p/4 \rfloor$. Moreover if $p \equiv_4 3$ then

$$c_m \equiv_{p^2} \frac{p}{2} \Gamma_p^4\left(\frac{1}{4}\right). \quad (21)$$

Proof. We start with (21). Since $p \equiv_4 3$, we have that $m = (p-3)/4$ and

$$\begin{aligned} c_m &= \frac{(-1)^m (2m+p)! (m!)^3}{(3m+1)! ((2m+1)!)^3} \equiv_{p^2} \frac{(-1)^{m+1} p (m!)^3}{2(3m+2)! ((2m+1)!)^2} \\ &= \frac{(-1)^{m+1} p \Gamma_p^3\left(\frac{p+1}{4}\right)}{2\Gamma_p\left(\frac{3p+3}{4}\right) \Gamma_p^2\left(\frac{p+1}{2}\right)} \equiv_{p^2} \frac{(-1)^{m+1} p \Gamma_p^3\left(\frac{1}{4}\right)}{2\Gamma_p\left(\frac{3}{4}\right) \Gamma_p^2\left(\frac{1}{2}\right)} \equiv_{p^2} \frac{p}{2} \Gamma_p^4\left(\frac{1}{4}\right) \end{aligned}$$

where, by (19),

$$\Gamma_p^2\left(\frac{1}{2}\right) = (-1)^{\frac{p+1}{2}} = 1, \quad \text{and} \quad \Gamma_p\left(\frac{1}{4}\right) \Gamma_p\left(\frac{3}{4}\right) = (-1)^{\frac{p+1}{4}} = (-1)^{m+1}. \quad (22)$$

As regards (20), we consider only the case $p \equiv_4 3$ since the other case can be handled similarly. Then

$$\frac{\binom{2m}{m}^2}{16^m} = \left(\frac{\binom{\frac{1}{2}}{m}}{\binom{1}{m}} \right)^2 = \frac{\Gamma_p^2(1)}{\Gamma_p^2(\frac{1}{2})} \cdot \frac{\Gamma_p^2(\frac{1}{2} + m)}{\Gamma_p^2(1 + m)} = \frac{\Gamma_p^2(-\frac{1}{4} + \frac{p}{4})}{\Gamma_p^2(\frac{1}{4} + \frac{p}{4})}.$$

By (18) and by [7, Lemma 2.4],

$$\Gamma_p \left(-\frac{1}{4} + \frac{p}{4} \right) \equiv_{p^2} \Gamma_p \left(-\frac{1}{4} \right) \left(1 + (G_1(1) + H_{3m+1}) \frac{p}{4} \right),$$

and

$$\Gamma_p \left(\frac{1}{4} + \frac{p}{4} \right) \equiv_{p^2} \Gamma_p \left(\frac{1}{4} \right) \left(1 + (G_1(1) + H_m) \frac{p}{4} \right).$$

Therefore, since $\Gamma_p(-\frac{1}{4}) = 4\Gamma_p(-\frac{3}{4})$,

$$\frac{\binom{2m}{m}^2}{16^m} \equiv_{p^2} \frac{\Gamma_p^2(-\frac{1}{4})}{\Gamma_p^2(\frac{1}{4})} \cdot \left(1 + (H_{3m+1} - H_m) \frac{p}{2} \right) \equiv_{p^2} 16\Gamma_p^{-4} \left(\frac{1}{4} \right) (1 + 2p)$$

where we also used (22) and

$$H_{3m+1} = H_{p-1} - \sum_{j=1}^{m+1} \frac{1}{p-j} \equiv_p H_m + \frac{4}{3p+1} \equiv_p H_m + 4.$$

□

5. CONGRUENCES FOR THE TRUNCATED SERIES - MAIN RESULTS

Theorem 2. *For any prime $p > 3$,*

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_k}{64^k} \equiv_{p^2} \begin{cases} \Gamma_p^4(\frac{1}{4}) \cdot (2q_p(2) - pq_p^2(2)) & \text{if } p \equiv_4 1, \\ -\frac{p}{12} \Gamma_p^4(\frac{1}{4}) & \text{if } p \equiv_4 3, \end{cases} \quad (23)$$

and

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_k^{(2)}}{64^k} \equiv_{p^2} \begin{cases} -\Gamma_p^4(\frac{1}{4}) \cdot (4E_{p-3} - 2E_{2p-4}) & \text{if } p \equiv_4 1, \\ -\frac{1}{4} \Gamma_p^4(\frac{1}{4}) & \text{if } p \equiv_4 3. \end{cases} \quad (24)$$

Proof. For (24), if $p \equiv_4 1$ then $n = (p-1)/2$ is even and we use (5) and (15). Finally we use (20). If $p \equiv_4 3$ then $n = (p-1)/2 = 2m+1$ is odd. We have

$$\frac{\binom{2k}{k}}{(-4)^k \binom{n}{k}} = \prod_{j=0}^{k-1} \left(1 - \frac{p}{2j+1} \right)^{-1} \equiv_{p^2} 1 - \frac{p}{2} \sum_{j=0}^{k-1} \frac{1}{n-j} = 1 - \frac{p}{2} (H_n - H_{n-k})$$

and therefore

$$\frac{1}{4^k} \binom{2k}{k} \equiv_{p^2} (-1)^k \binom{n}{k} \left(1 - \frac{p}{2} (H_n - H_{n-k}) \right). \quad (25)$$

Thus, by (12), (16), and (21),

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_k}{64^k} &\equiv_{p^2} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 \left(1 - \frac{3p}{2} (H_n - H_{n-k})\right) H_k \\ &\equiv_{p^2} \left(1 - \frac{3p}{2} H_n\right) \left(-\frac{c_m}{6}\right) \equiv_{p^2} -\frac{p}{12} \Gamma_p^4 \left(\frac{1}{4}\right). \end{aligned}$$

As regards (24), we use (5) and (15) with $n = (p-1)/2$. Then we apply (20) and (7). \square

Theorem 3. For any prime $p > 3$,

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{O_k}{64^k} \equiv_{p^2} \begin{cases} 0 & \text{if } p \equiv_4 1, \\ -\frac{p}{12} \Gamma_p^4 \left(\frac{1}{4}\right) & \text{if } p \equiv_4 3, \end{cases} \quad (26)$$

and

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{O_k^{(2)}}{64^k} \equiv_p \begin{cases} \frac{1}{2} \Gamma_p^4 \left(\frac{1}{4}\right) E_{p-3} & \text{if } p \equiv_4 1, \\ -\frac{1}{16} \Gamma_p^4 \left(\frac{1}{4}\right) & \text{if } p \equiv_4 3. \end{cases} \quad (27)$$

Proof. If $n = (p-1)/2$ is even then by (5), (16) and (17),

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{O_k}{64^k} = \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \left(H_{2k} - \frac{H_k}{2}\right) \equiv_{p^2} \binom{n}{\frac{n}{2}}^2 \frac{H_n}{2 \cdot 4^n} - \frac{1}{2} \binom{n}{\frac{n}{2}}^2 \frac{H_n}{4^n} \equiv_{p^2} 0.$$

Assume now that $n = (p-1)/2 = 2m+1$ is odd. We have that

$$H_{2(n-k)} = H_{p-1} - \sum_{j=1}^{2k} \frac{1}{p-j} \equiv_{p^2} H_{2k} + p H_{2k}^{(2)} \equiv_{p^2} H_{2k} + \frac{p}{4} (H_k^{(2)} - H_{n-k}^{(2)}).$$

Hence

$$\begin{aligned} \sum_{k=1}^n (-1)^k \binom{n}{k}^3 H_{2k} &= - \sum_{k=0}^n (-1)^k \binom{n}{n-k}^3 H_{2(n-k)} \\ &\equiv_{p^2} - \sum_{k=1}^n (-1)^k \binom{n}{k}^3 H_{2k} - \frac{p}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 (H_k^{(2)} - H_{n-k}^{(2)}) \end{aligned}$$

which implies

$$\sum_{k=1}^n (-1)^k \binom{n}{k}^3 H_{2k} \equiv_{p^2} -\frac{p}{8} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 (H_k^{(2)} - H_{n-k}^{(2)}) = -\frac{p}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_k^{(2)}. \quad (28)$$

Moreover

$$H_{2k} = \frac{H_k}{2} + \sum_{j=0}^{k-1} \frac{1}{2j+1} \equiv_p \frac{1}{2} \left(H_k - \sum_{j=0}^{k-1} \frac{1}{n-j} \right) \equiv_p \frac{1}{2} (H_k + H_{n-k} - H_n). \quad (29)$$

Consequently by (25), (28), (29),

$$\begin{aligned}
\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_{2k}}{64^k} &\equiv_{p^2} \sum_{k=1}^n (-1)^k \binom{n}{k}^3 \left(1 - \frac{3p}{2} (H_n - H_{n-k})\right) H_{2k} \\
&\equiv_{p^2} \sum_{k=1}^n (-1)^k \binom{n}{k}^3 H_{2k} - \frac{3p}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 (H_n - H_{n-k}) (H_k + H_{n-k} - H_n) \\
&\equiv_{p^2} -\frac{p}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_k^{(2)} - \frac{3p}{2} H_n \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_k \\
&\quad + \frac{3p}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_{n-k}^2 - \frac{3p}{4} H_n \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_{n-k} \\
&\equiv_{p^2} -\frac{p}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 \left(3H_k^2 + H_k^{(2)}\right) \equiv_{p^2} -\frac{cm}{4} \equiv_{p^2} -\frac{p}{8} \Gamma_p^4 \left(\frac{1}{4}\right) \quad (30)
\end{aligned}$$

where in the last step we used (13), (14), and (21) (note that $m < 2m + 1 < 3m + 2 < p < 6m + 4 < 2p$). Finally, by (23),

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{O_k}{64^k} = \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \left(H_{2k} - \frac{H_k}{2}\right) \equiv_{p^2} -\frac{p}{12} \Gamma_p^4 \left(\frac{1}{4}\right)$$

and the proof of (26) is complete.

As regards (27), we have by (6)

$$O_k^{(2)} = \sum_{j=0}^{k-1} \frac{1}{(2j+1)^2} \equiv_p \sum_{j=0}^{k-1} \frac{1}{4(n-j)^2} = \frac{H_n^{(2)} - H_{n-k}^{(2)}}{4} \equiv_p -\frac{H_{n-k}^{(2)}}{4} \quad (31)$$

where $n = (p-1)/2$. Then, by (25) and (31),

$$\begin{aligned}
\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{O_k^{(2)}}{64^k} &\equiv_p -\frac{1}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_{n-k}^{(2)} = \frac{(-1)^{n+1}}{4} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 H_k^{(2)} \\
&\equiv_p \frac{(-1)^{n+1}}{4} \sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_k^{(2)}}{64^k}
\end{aligned}$$

and the desired result follows from (24). □

Remark 4. By (23), (26), and (2) for any prime $p > 3$

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_{2k} - H_k}{64^k} \equiv_p q_p(2) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k}$$

which is a particular case of [6, Corollary 2]. Moreover by (23) and (30), if $p \equiv_4 3$ then

$$\sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_k}{64^k} \equiv_p \sum_{k=1}^{p-1} \binom{2k}{k}^3 \frac{H_{2k}}{64^k} \equiv_p 0$$

which appears in [18].

6. CODA

In this final section we present a few more results with the same flavor related to ${}_2F_1(1/2)$, ${}_4F_3(-1)$ and ${}_6F_5(-1)$.

By Bailey's theorem [1, p.11],

$${}_2F_1 \left[\begin{matrix} a, 1-a \\ c \end{matrix}; \frac{1}{2} \right] = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{a+c}{2})\Gamma(\frac{1-a+c}{2})} \quad (32)$$

it follows that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{32^k} = {}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \frac{1}{2} \right] = \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}. \quad (33)$$

Moreover, it has been proved (see for example [16, Corollary 2.2] and [7, (1.4)])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv_{p^2} \begin{cases} (-1)^{\frac{p+1}{2}} \Gamma_p(\frac{1}{2}) \Gamma_p(\frac{1}{4})^2 & \text{if } p \equiv_4 1, \\ 0 & \text{if } p \equiv_4 3. \end{cases} \quad (34)$$

Note that, by Clausen's Formula and its truncated version [3, Lemma18], equations (1), (33), and congruences (2), (34) satisfy the following connecting relationships,

$$\left(\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{32^k} \right)^2 = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{64^k} \quad \text{and} \quad \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \right)^2 \equiv_{p^2} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k}.$$

Now by letting $a = 1/2$ in (32), then

$$\sum_{k=1}^{\infty} \binom{2k}{k}^2 \frac{H_k}{32^k} = \frac{\partial}{\partial c} \left({}_2F_1 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ c \end{matrix}; \frac{1}{2} \right] \right) \Big|_{c=1} = \frac{\partial}{\partial c} \left(\frac{2^{1-c} \sqrt{\pi} \Gamma(c)}{\Gamma^2(\frac{1}{4} + \frac{c}{2})} \right) \Big|_{c=1} = \frac{\sqrt{\pi} (\pi - 4 \ln 2)}{2\Gamma^2(\frac{3}{4})}.$$

The following result yields a p -adic analog of the above series.

Theorem 5. *For any prime $p > 3$,*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{H_k}{32^k} \equiv_{p^2} \begin{cases} \Gamma_p(\frac{1}{2}) \Gamma_p^2(\frac{1}{4}) \cdot (2q_p(2) - pq_p^2(2)) & \text{if } p \equiv_4 1, \\ \frac{1}{2} \Gamma_p(\frac{1}{2}) \Gamma_p^2(\frac{1}{4}) & \text{if } p \equiv_4 3. \end{cases} \quad (35)$$

Proof. It suffices to use the identity,

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{H_k}{(-2)^k} = \begin{cases} \left(\frac{n}{2}\right)^2 \cdot \frac{(-1)^{\frac{n}{2}} H_n}{2^n} & \text{if } n \equiv_2 0, \\ \left(\frac{n-1}{2}\right)^{-1} \cdot \frac{(-1)^{\frac{n+1}{2}} 2^{n-1}}{n} & \text{if } n \equiv_2 1, \end{cases}$$

then the verification of congruence (35) can be carried out along the lines of the proof (23). The interested reader may fill in the necessary details. \square

By a couple of formulas which appear in [1, (2) and (3) p.28],

$${}_4F_3 \left[\begin{matrix} a, 1 + \frac{a}{2}, d, e \\ \frac{a}{2}, 1 + a - d, 1 + a - e \end{matrix}; -1 \right] = \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}, \quad (36)$$

$${}_6F_5 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, d, e \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e \end{matrix}; -1 \right] \quad (37)$$

$$= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)} {}_3F_2 \left[\begin{matrix} 1 + a - b - c, d, e \\ 1 + a - b, 1 + a - c \end{matrix}; 1 \right], \quad (38)$$

we have the series labeled (B.1) and (A.1) in [21]

$$\begin{aligned} \sum_{k=0}^{\infty} (4k+1) \frac{\binom{2k}{k}^3}{64^k} (-1)^k &= {}_4F_3 \left[\begin{matrix} \frac{1}{2}, \frac{5}{4}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1 \end{matrix}; -1 \right] = \frac{2}{\pi}, \\ \sum_{k=0}^{\infty} (4k+1) \frac{\binom{2k}{k}^5}{1024^k} (-1)^k &= {}_6F_5 \left[\begin{matrix} \frac{1}{2}, \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1 \end{matrix}; -1 \right] = \frac{2}{\Gamma^4\left(\frac{3}{4}\right)}. \end{aligned}$$

Moreover, it has been shown that the p -analogs (B.2) and (A.2) in [21] hold for any prime $p > 3$ (see [10] and [8]),

$$\begin{aligned} \sum_{k=0}^{p-1} (4k+1) \frac{\binom{2k}{k}^3}{64^k} (-1)^k &\equiv_{p^3} (-1)^{\frac{p-1}{2}} p \\ \sum_{k=0}^{p-1} (4k+1) \frac{\binom{2k}{k}^5}{1024^k} (-1)^k &\equiv_{p^3} \begin{cases} -\frac{p}{\Gamma_p^4\left(\frac{3}{4}\right)} & \text{if } p \equiv_4 1, \\ 0 & \text{if } p \equiv_4 3, \end{cases} \end{aligned}$$

Recently Guillera proved this elegant Ramanujan-type formula involving harmonic numbers [4, (32)],

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{64^k} (2 - 3(4k+1)H_k) (-1)^k = \frac{12 \ln 2}{\pi} \quad (39)$$

The above evaluation can be established by noting that the left-hand side is equal to

$${}_2F_3 \left[\begin{matrix} \frac{1}{2}, \frac{5}{4}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1 \end{matrix}; -1 \right] + \frac{\partial}{\partial a} \left({}_4F_3 \left[\begin{matrix} a, 1 + \frac{a}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{a}{2}, \frac{1}{2} + a, \frac{1}{2} + a \end{matrix}; -1 \right] \right) \Big|_{a=\frac{1}{2}} - \frac{\partial}{\partial e} \left({}_4F_3 \left[\begin{matrix} \frac{1}{2}, \frac{5}{4}, \frac{1}{2}, e \\ \frac{1}{4}, 1, \frac{3}{2} - e \end{matrix}; -1 \right] \right) \Big|_{e=\frac{1}{2}}.$$

Then by using (36) we obtain the right-hand side. In a similar way, from (37), we can get

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^5}{1024^k} (2 - 5(4k+1)H_k) (-1)^k = \frac{4(15 \ln(2) - 2\pi)}{3\Gamma^4\left(\frac{3}{4}\right)}. \quad (40)$$

The infinite series (39) and (40) have p -adic analogs which are given in the next result.

Theorem 6. *For any prime $p > 3$,*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} (2 - 3(4k+1)H_k) (-1)^k \equiv_{p^2} (-1)^{\frac{p-1}{2}} (2 + 6pq_p(2)), \quad (41)$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^4}{256^k} (2 - 4(4k+1)H_k) \equiv_{p^2} 2 + 12pq_p(2), \quad (42)$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^5}{1024^k} (2 - 5(4k+1)H_k) (-1)^k \equiv_{p^2} \begin{cases} -(2 + 10pq_p(2))\Gamma_p^4\left(\frac{1}{4}\right) & \text{if } p \equiv_4 1, \\ 0 & \text{if } p \equiv_4 3. \end{cases} \quad (43)$$

Proof. Let $n = \frac{p-1}{2}$, then, by (25), the left-hand side of (41) is congruent modulo p^2 to

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k}^3 \left(1 - \frac{3p}{2} (H_n - H_{n-k}) \right) (2 + 3(2n - 4k - p)H_k) \\ & \equiv_{p^2} (2 - 3pH_n) \sum_{k=0}^n \binom{n}{k}^3 (1 + 3(n - 2k)H_k) + 3p \sum_{k=0}^n \binom{n}{k}^3 H_{n-k} \\ & \quad + 9p \sum_{k=0}^n \binom{n}{k}^3 (n - 2k)H_k H_{n-k} - 3p \sum_{k=0}^n \binom{n}{k}^3 H_k \\ & \equiv_{p^2} (-1)^n (2 - 3pH_n) \equiv_{p^2} (-1)^{\frac{p-1}{2}} (2 + 6pq_p(2)) \end{aligned}$$

where we used (4) and the identities

$$\sum_{k=0}^n \binom{n}{k}^3 (1 + 3(n - 2k)) H_k = (-1)^n \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k}^3 (n - 2k) H_k H_{n-k} = 0.$$

The first one follows from [11, (3)], whereas the second one follows by replacing k with $(n - k)$.

Congruences (42) and (43) can be obtained in a similar way by using the identities

$$\sum_{k=0}^n \binom{n}{k}^4 (1 + 4(n - 2k)) H_k = (-1)^n \binom{2n}{n},$$

$$\sum_{k=0}^n \binom{n}{k}^5 (1 + 5(n - 2k)) H_k = (-1)^n \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k},$$

which are equivalent to [11, (4)] and [11, (5)] respectively. \square

REFERENCES

- [1] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935.
- [2] X. Chen and W. Chu, *Dixons ${}_3F_2(1)$ and identities involving harmonic numbers and the Riemann zeta function*, Discrete Math. **310** (2010), 83–91.
- [3] S. Chisholm, A. Deines, L. Long, G. Nebe and H. Swisher, *p-adic analogues of Ramanujan type formulas for $1/\pi$* , Mathematics **32** (2013), 9–30.
- [4] J. Guillera, *More hypergeometric identities related to Ramanujan-type series*, Ramanujan J. **32** (2013), 5–22.
- [5] H.-T. Jin and D.K. Du, *Abel’s lemma and identities on harmonic numbers*, Integers **15** (2015), #A22.
- [6] J. Kibebek, L. Long, K. Moss, B. Sheller and H. Yuan, *Supercongruences and complex multiplication*, J. Number Theory **164** (2016), 166–178.
- [7] Ji-Cai Liu, *Generalized Rodriguez-Villegas supercongruences involving p-adic Gamma functions*, arXiv:1611.07686 (november 2016).
- [8] L. Long, *Hypergeometric evaluations identities and supercongruences*, Pac. J. Math. **249** (2011), 405–418.
- [9] L. Long and R. Ramakrishna, *Some supercongruences occurring in truncated hypergeometric series*, Adv. Math. **290** (2016), 773–808.
- [10] D. McCarthy and R. Osburn, *A p-adic analogue of a formula of Ramanujan* Arch. Math. **91** (2008), 492–504.
- [11] P. Paule and C. Schneider, *Computer proofs of a new family of harmonic number identities*, Adv. in Appl. Math. **31** (2003), 359–378.
- [12] H. Prodinger, *Human proofs of identities by Osburn and Schneider*, Integers **8** (2008), #A10.
- [13] A. Robert, *A course in p-adic Analysis*, Springer-Verlag, New York, 2000.
- [14] Z.-H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), 193–223.
- [15] Z.-H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory **128** (2008), 280–312.
- [16] Z.-H. Sun, *Generalized Legendre polynomials and related supercongruences*, J. Number Theory **143** (2014), 293–319.
- [17] Z.-W. Sun, *New congruences involving harmonic numbers*, arXiv:1407.8465 (august 2014).
- [18] Z.-W. Sun, *Determining x or $y \bmod p^2$ with $p = x^2 + dy^2$* , arXiv:1210.5237 (june 2015).
- [19] Z.-W. Sun, *A new series for π^3 and related congruences*, Int. J. Math. **26** (2015), 1550055.
- [20] R. Tauraso, *Congruences involving alternating multiple harmonic sum*, Elec. J. Comb. **17** (2010), #R16.
- [21] L. Van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series, p-adic functional analysis* (Nijmegen, 1996), 223–236, Lecture Notes in Pure and Appl. Math., 192, Dekker, New York, 1997.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA “TOR VERGATA”, VIA DELLA RICERCA
SCIENTIFICA, 00133 ROMA, ITALY

E-mail address: `tauraso@mat.uniroma2.it`